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A Parallel Plate Waveguide With Only One Conductor

by
P. L. Overfelt
Physics Division
Research and Intelligence Department

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NAVAL AIR WARFARE CENTER WEAPONS DIVISION
CHINA LAKE, CA 93555-6100

Naval Air Warfare Center Weapons Division

FOREWORD

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Approved by
R. A. NISSAN, *Head*
Research and Intelligence Department
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M. W. WINTER
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(U) The parallel plate waveguide consisting of two perfectly conducting plates enclosing a homogeneous, isotropic medium is very well known. In this report, a parallel plate waveguide containing an inhomogeneous permittivity function of specific form is considered. Assuming an exponential dielectric function of position, it is possible to eliminate one of the waveguide conducting plates and still keep the electromagnetic field closely confined to the remaining plate and held within the inhomogeneous region.

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1.0 INTRODUCTION

The parallel plate waveguide consisting of two perfectly conducting plates enclosing a homogeneous isotropic medium is very well known. When it is assumed that the z direction is the direction of propagation and the plates are located along the x direction (at x = 0 and x = a) (see Figure 1), the mode solutions along the x direction are sinusoidal, either $\sin\left(\frac{n\pi x}{a}\right)$ or $\cos\left(\frac{n\pi x}{a}\right)$, where a is the plate separation and n can be any integer. These sinusoidal functions are periodic, and higher order modes have nodes that can segment the plate separation distance into equidistant parts (Reference 1). Since n can be any integer, the number of such modes at a given frequency can be denumerably infinite.

In this report, a parallel plate waveguide containing an inhomogeneous dielectric (or permittivity) function of specific form is considered. It is found that by assuming an exponential dielectric function of position, it is possible to remove one of the waveguide conducting plates and still keep the fields closely confined to the remaining plate and held within the inhomogeneous region.

2.0 INHOMOGENEOUS PERMITTIVITY IN MAXWELL'S EQUATIONS

Assuming that the propagation direction is along the z-axis and that the z-direction dependence and the time dependence are given by $e^{i(k_z z-\omega t)}$, we start with Maxwell's equations in the form

$$\begin{aligned}&\nabla\times\vec{E}=i\omega\vec{B};\quad \nabla\cdot\vec{B}=0\\ &\qquad\text{and}\\ &\nabla\times\vec{H}=-i\omega\vec{D};\quad \nabla\cdot\vec{D}=0\end{aligned}\tag{1}$$

Assuming that we have a permittivity function depending on a single transverse coordinate and that the permeability is constant, let

$$\vec{B}=\mu_0\mu\vec{H}\tag{2a}$$

and

$$\vec{D}=\varepsilon_0\varepsilon(x)\vec{E}\tag{2b}$$

Furthermore, assuming an infinite flat slab type of geometry (see Figure 1) with $\frac{\partial}{\partial y} = 0$, Maxwell's equations in component form become

$$\frac{\partial E_y}{\partial z} = -i\omega\mu_0\mu H_x \tag{3a}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega\mu_0\mu H_y \tag{3b}$$

$$\frac{\partial E_y}{\partial x} = i\omega\mu_0\mu H_z \tag{3c}$$

and

$$\frac{\partial H_y}{\partial z} = i\omega\varepsilon_0\varepsilon(x) E_x \tag{4a}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0\varepsilon(x) E_y \tag{4b}$$

$$\frac{\partial H_y}{\partial x} = -i\omega\varepsilon_0\varepsilon(x) E_z \tag{4c}$$

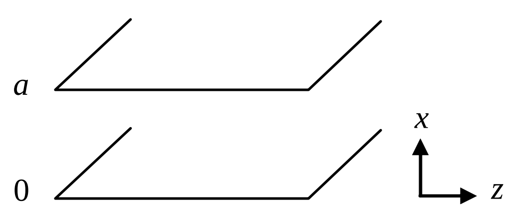


FIGURE 1. Parallel Conducting Plates at $x = 0$ and $x = a$.

Taking the derivative with respect to *z* of Equation 3a and the derivative with respect to *x* of Equation 3c and substituting the result into Equation 4b, Equation 4b becomes

$$\frac{d^2E_y}{dx^2}+\left[k_0^2\mu\varepsilon\left(x\right)-k_z^2\right]E_y=0 \tag{5}$$

It is assumed throughout that the total field components are given by $E_j^{tot}\left(x,z\right)=E_j\left(x\right)e^{i\left(k_z z-\omega t\right)}$; *j* = *x*,*y*,*z*, and $H_j^{tot}\left(x,z\right)=H_j\left(x\right)e^{i\left(k_z z-\omega t\right)}$; *j* = *x*,*y*,*z*.

Thus in the above components, only $E_j\left(x\right)$ and $H_j\left(x\right)$ must be determined. This in turn leads to ordinary differential equations (ODEs) that need to be solved as opposed to partial differential equations (PDEs).

Equation 5 is formally equivalent to a one-dimensional (1D) Schrodinger equation with k_z playing the role of an eigenvalue and $\varepsilon\left(x\right)$, the potential. Once we have E_y, from Equations 3a and 3c we have H_z and H_x. These are transverse electric (TE) modes with respect to *z*. In general, for inhomogeneous media it is not possible to split modes into TE and transverse magnetic (TM) classes, but when the permittivity function is transversely dependent on a single variable, it is possible to make this split (References 2 through 5).

Returning to Equations 3 and 4, we can obtain an ODE for H_y also. Taking the derivative of Equation 4a with respect to *z* and the derivative of Equation 4c with respect to *x* and substituting the result into Equation 3b gives

$$\frac{d^2H_y}{dx^2}-\frac{\varepsilon'\left(x\right)}{\varepsilon\left(x\right)}\frac{dH_y}{dx}+\left[k_0^2\mu\varepsilon\left(x\right)-k_z^2\right]H_y=0 \tag{6}$$

where $\varepsilon'\left(x\right)=\frac{d\varepsilon\left(x\right)}{dx}$.

Because we chose to have the permittivity be a function of *x* but not the permeability, our equations for E_y and H_y are not symmetric. However from Equations 3 and 4, if Equation 6 can be solved for H_y, then both E_x and E_z follow. Thus these are the TM modes with respect to *z*, the propagation direction.

3.0 EXPONENTIAL PERMITTIVITY

Let the permittivity be given by

$$\varepsilon(x) = be^{-\delta x} \tag{7}$$

where *b* is often called the strength or intensity of the function and δ is called the range.

3.1 TE CASE

Substituting Equation 7 into Equation 5, we get

$$\frac{d^2 E_y}{dx^2} + \left[k_0^2 \mu b e^{-\delta x} - k_z^2 \right] E_y = 0 \tag{8}$$

Solving Equation 8, we obtain for the general solution (Reference 6)

$$E_y(x) = c_1 \Gamma\left(1 + \frac{2k_z}{\delta}\right) J_{\frac{2k_z}{\delta}}\left(\frac{2k_0\sqrt{\mu b}}{\delta} e^{-\delta x/2}\right) + c_2 \Gamma\left(1 - \frac{2k_z}{\delta}\right) J_{\frac{-2k_z}{\delta}}\left(\frac{2k_0\sqrt{\mu b}}{\delta} e^{-\delta x/2}\right) \tag{9}$$

The *J*'s are Bessel functions of the first kind. Γ denotes the Euler gamma function.

We are interested in finite solutions for a conducting parallel plate geometry as in Figure 1. For now, let us assume that one plate is at $x = 0$ while the other is at $x = \infty$. By doing this, we see that the first solution of Equation 9 will go to zero as $x \to \infty$ while the second solution will have a singularity there.

Thus we throw away the second solution by setting $c_2 = 0$ in Equation 9. The electric field will be taken as

$$E_y = c_1' J_{\frac{2k_z}{\delta}}\left(\frac{2k_0\sqrt{\mu b}}{\delta} e^{-\delta x/2}\right) \tag{10}$$

with $k_0 = \frac{\omega}{c}$, the free space wave number, and

$$c_1' = c_1 \Gamma\left(1 + \frac{2k_z}{\delta}\right) \tag{11}$$

Note that the gamma function in Equation 11 is a function of k_z, the eigenvalue, and it will change for different modes. To give Equation 10 a dimensionless form, we introduce $\beta = \frac{k_z}{k_0}$ where β is the normalized longitudinal propagation constant, and by substitution into Equation 10 we have

$$E_y = c_1' J_{\frac{2k_0\beta}{\delta}}\left(\frac{2k_0\sqrt{\mu b}}{\delta} e^{-\delta x/2}\right) \tag{12}$$

and

$$c_1' = c_1 \Gamma\left(1 + \frac{2k_0\beta}{\delta}\right) \tag{13}$$

β is dimensionless, and thus as long as δ has the same units as k_0 (e.g., k_0 and δ must both have units of m^{-1} if x is in meters), Equation 12 is now dimensionless.

Now that E_y is known, we can obtain the magnetic field components from Equations 3a and c

$$H_x = \frac{-k_z}{\omega\mu_0\mu} E_y \tag{14a}$$

and

$$H_z = \frac{1}{i\omega\mu_0\mu}\frac{dE_y}{dx} \tag{14b}$$

Thus H_x is proportional to E_y and by setting $\nu = \frac{2k_0\beta}{\delta}, \eta = \frac{2k_0\sqrt{\mu b}}{\delta}$, H_z can be written as

$$H_z = \frac{-c_1'\sqrt{\frac{b}{\mu}}}{i2\mu_0 c} e^{-\delta x/2}\left[J_{\nu-1}\left(\eta e^{-\delta x/2}\right) - J_{\nu+1}\left(\eta e^{-\delta x/2}\right)\right] \tag{15}$$

Equations 12, 14a, and 15 give the TE (to $\hat{z}$) modes of this inhomogeneous geometry.

At this point, the boundary conditions must be considered. The geometry (so far) consists of two perfectly conducting parallel plates at x = 0 and x = ∞. The boundary conditions are tangential $\vec{E}$ is zero on x = 0, ∞ and normal $\vec{B}$ is zero on x = 0, ∞. Also $\frac{dH_z}{dn} = 0$ on x = 0, ∞ where n refers to the normal to the plates. These imply

$$E_y = 0, H_x = 0, \text{and } \frac{dH_z}{dx} = 0 \text{ on } x = 0, \infty \tag{16}$$

Since E_y and H_x are proportional, if one of them satisfies its boundary condition, the other will also. Bear in mind that

$$\frac{dH_z}{dx} = \frac{1}{i\omega\mu_0\mu}\frac{d^2E_y}{dx^2} \tag{17}$$

We see from Equation 8, if $E_y = 0$ at some point, $\frac{d^2E_y}{dx^2} = 0$ there also. Thus all three boundary conditions in Equation 16 are satisfied when $E_y = 0$ is satisfied.

Now E_y must be 0 at x = 0 and at x = ∞. Using Equation 12, E_y is zero at infinity automatically without restrictions on either the argument or the order parameters. Thus from Equation 12, we are left with the boundary condition (BC) at x = 0:

$$BC_1 = J_{\frac{2k_0\beta}{\delta}}\left(\frac{2k_0\sqrt{\mu b}}{\delta}\right) = 0 \tag{18}$$

where $k_0 = \frac{2\pi f_0}{c}$ and f_0 is the linear frequency.

For now it is assumed that k_0, δ, b, and μ are all real and positive. This restriction can be relaxed later. In this case, β is not only the propagation constant, it is also the eigenvalue of Equation 18. So the value(s) of β that causes a zero of the Bessel function in Equation 18 is found given the frequency as well as b, μ, and δ.

In order for Equation 18 to have a real solution, β, at all, it is necessary that the strength of the dielectric function be greater than a certain limit. This limit results from the fact that the first zero of $J_\nu(\eta)$ moves to smaller values of η when ν is decreased (Reference 7). When $\beta = 0$, the order of the Bessel function in Equation 18 is also zero,

and the lowest zero of the zeroth order Bessel function is 2.4048. Thus Equation 18 has a real solution only if the argument of Equation 18

$$\frac{2k_0\sqrt{\mu b}}{\delta} > 2.4048 \tag{19}$$

or

$$b > \frac{1.4457}{\mu\left(\frac{k_0^2}{\delta^2}\right)} \tag{20}$$

From Equation 20, at lower frequencies *b* must be large and *b* can decrease as the frequency increases.

3.2 TM CASE

For the TM modes, we must solve a slightly more complicated equation than the one used for the TE case. Now we must solve Equation 6 when Equation 7 is substituted in. Note that Equation 6 is no longer in the form of a 1D Schrodinger equation by virtue of the first-order derivative in *x*. The equation to solve is

$$\frac{d^2H_y}{dx^2} + \delta\frac{dH_y}{dx} + \left[k_0^2\mu b e^{-\delta x} - k_z^2\right]H_y = 0 \tag{21}$$

Using $\eta = \frac{2k_0\sqrt{\mu b}}{\delta}$ as before and

$$\sigma = \sqrt{\left(\frac{2k_0\beta}{\delta}\right)^2 + 1} = \sqrt{\nu^2 + 1} \tag{22}$$

the general solution to Equation 21 is (Reference 6)

$$H_y = \frac{\eta}{2}e^{-\delta x/2}\left\{d_1{}'J_\sigma\left(\eta e^{-\delta x/2}\right) + d_2{}'J_{-\sigma}\left(\eta e^{-\delta x/2}\right)\right\} \tag{23}$$

with

$$d_1{}' = d_1\Gamma(1+\sigma),\ d_2{}' = d_2\Gamma(1-\sigma)$$

where d_1 and d_2 are general constants. As previously, we set $d_2 = 0$ and eliminate the Bessel function of negative order.

Thus the solution we will use is

$$H_y = \frac{\eta}{2} d_1' e^{-\delta x/2} J_\sigma \left(\eta e^{-\delta x/2} \right) \tag{24}$$

At this point, the TM case must proceed a little differently from the TE case. Although we know that

$$E_x = \frac{k_z}{\omega \varepsilon_0 \varepsilon(x)} H_y \tag{25a}$$

and

$$E_z = \frac{1}{-i\omega \varepsilon_0 \varepsilon(x)} \frac{dH_y}{dx} \tag{25b}$$

we see that we would be dividing both H_y and dH_y/dx components by $\varepsilon(x)$ which goes to zero (or at least becomes very small at some point). Thus using the electric displacement, $\vec{D}$, at this point, i.e.,

$$D_x = \frac{k_z}{\omega \varepsilon_0} H_y \tag{26a}$$

and

$$D_z = \frac{1}{-i\omega \varepsilon_0} \frac{dH_y}{dx} \tag{26b}$$

is a better choice.

Thus D_x is proportional to H_y, but neither of these components has a boundary condition associated with it on a perfect conductor. D_z can be written as

$$D_z = \frac{d_1' \delta \eta}{8i\omega \varepsilon_0} \left\{ \eta e^{-\delta x} \left[J_{\sigma-1}\left(\eta e^{-\delta x/2} \right) - J_{\sigma+1}\left(\eta e^{-\delta x/2} \right) \right] + 2e^{-\delta x/2} J_\sigma \left(\eta e^{-\delta x/2} \right) \right\} \tag{27}$$

Now the only boundary condition we must meet for the TM modes is $E_z = 0$ on x = 0, ∞. If $E_z = 0$ on these boundaries, then $D_z = 0$ on the boundaries also.

Thus Equation 27 reduces to (at x = 0)

$$BC_2 = \eta\left[J_{\sigma-1}(\eta) - J_{\sigma+1}(\eta)\right] + 2J_{\sigma}(\eta) = 0 \tag{28}$$

Since σ contains the eigenvalue, β, we solve BC_2 for the value(s) of β that cause Equation 28 to be zero. This is a slightly more difficult equation to solve than the one for the TE case.

4.0 NUMERICAL RESULTS

4.1 TE CASE

For the TE case, E_y is the only nonzero component of the electric field, and H_x and H_z are the corresponding nonzero magnetic field components. Since it has been shown in Section 3.0 that all boundary conditions are satisfied, all numerical results will focus on E_y alone.

Recall that μ is the relative (constant) permeability, b is the relative strength of the dielectric function, δ is the range of the dielectric function, and f_0 is the linear frequency. Because δ and k_0 have the same units (i.e., m^{-1} or cm^{-1}, etc.) and μ and b are dimensionless, we solve Equation 18 for β, the normalized propagation constant.

In Table 1, the ordered roots, β, of Equation 18 are given for $\delta = 1$, $\sqrt{\mu b} = 1$, as a function of frequency. β_1 is the eigenvalue for the lowest-order mode, β_2 is the eigenvalue for the next lowest-order mode, and so on. As the frequency increases, more and more higher-order modes can propagate assuming a fixed δ and $\sqrt{\mu b}$. Figure 2 shows a plot of frequency versus β also.

TABLE 1. Ordered TE Roots, β, of Equation 18 With $\delta = 1$, $\sqrt{\mu b} = 1$.

f_0 (GHz)	β_1	β_2	β_3	β_4	β_5	β_6	β_7
6	0.0282						
7	0.1206						
8	0.1937						
9	0.2531						
10	0.3026						
11	0.3446						
12	0.3808						
13	0.4124						
14	0.4400	0.0380					
15	0.4649	0.0796					
16	0.4870	0.1169					
17	0.5070	0.1814					
18	0.5251	0.1814					
19	0.5417	0.2088					
20	0.5569	0.2353					
24	0.6069	0.3205	0.0914				
36	0.6991	0.4782	0.3004	0.1457	0.0063		
94	0.8405	0.7223	0.6263	0.5421	0.4656	0.3948	0.3283

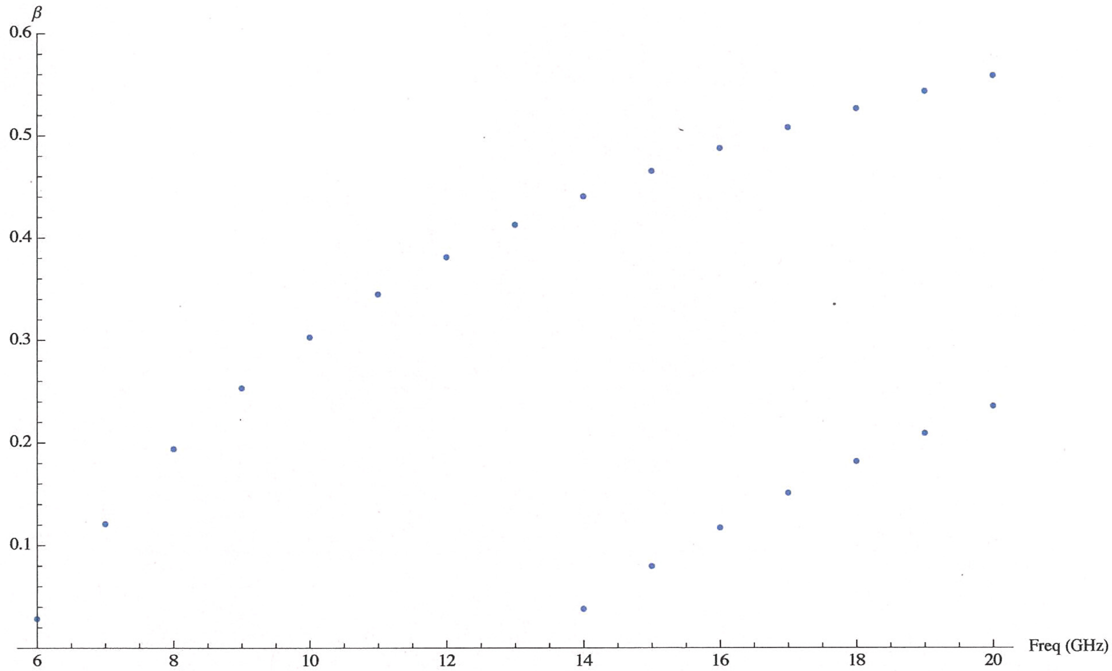
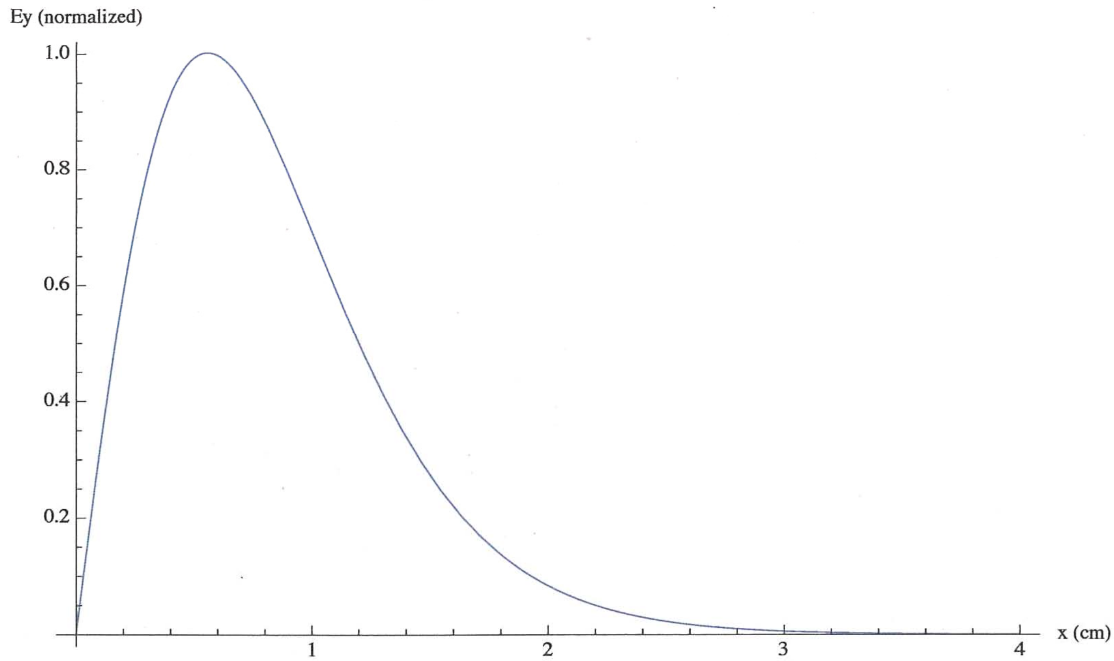
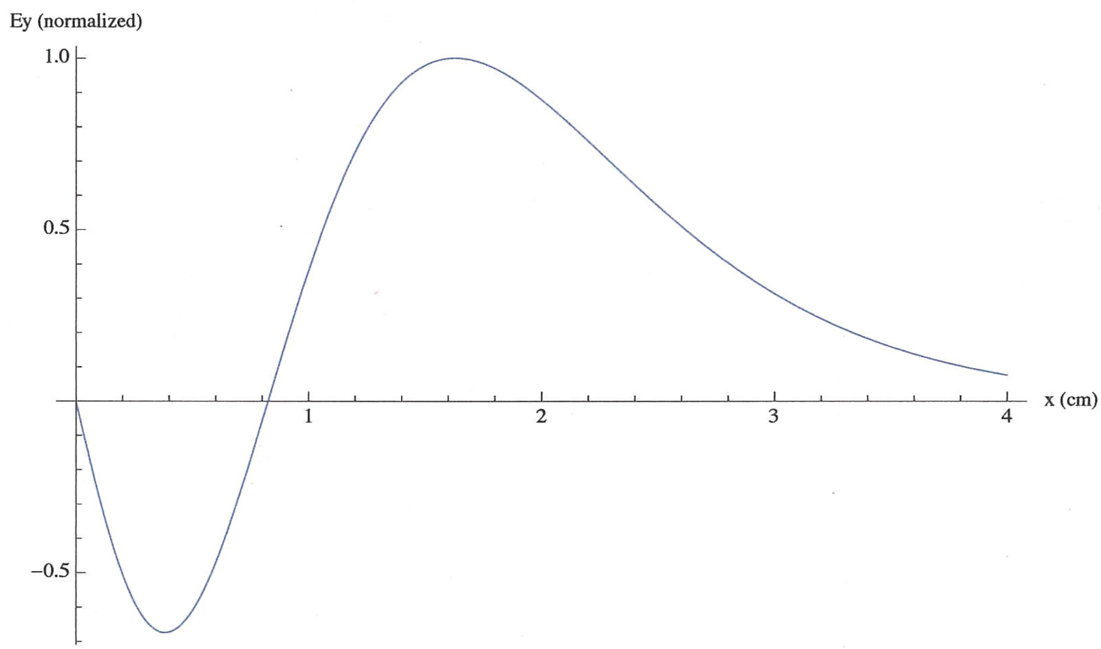


FIGURE 2. TE Longitudinal Propagation Constant, β,
Versus Frequency $(\mu = b = \delta = 1)$.

Figure 2 gives the dispersion curves of this parallel plate geometry with an inhomogeneous permittivity of exponential form. For a given δ, μ, and *b*, once the roots of Equation 18 are found at a given frequency, one can plot the normalized electric field versus *x* (*cm*) (*x* is the distance along the transverse direction between the plates). Notice that so far we have still assumed that the second plate is at $x = \infty$. Figures 3(a), (b), (c), and (d) show plots of normalized E_y versus *x* for $\delta = 1$, $\sqrt{\mu b} = 1$, and $f_0 = 24$ GHz. At 24 GHz, there are three real roots of Equation 18 (see Table 1). Using $\beta = \beta_1$ in E_y gives Figure 3(a), the lowest-order mode; using $\beta = \beta_2$ in E_y gives Figure 3(b), the second mode; and $\beta = \beta_3$ in E_y gives Figures 3(c) and 3(d), the third mode. Figure 3(d) is the same plot as Figure 3(c), only on an expanded scale. In Figure 3(a), note that E_y is highly concentrated near the conducting plate at $x = 0$ and trails off to zero as *x* increases. (The asymptotic forms of the fields are given in Appendix A.) The orthogonality property of different propagating modes at a given frequency is shown in Appendix B. Also the Poynting vector and the power flow along the inhomogeneous waveguide are derived in Appendix C.

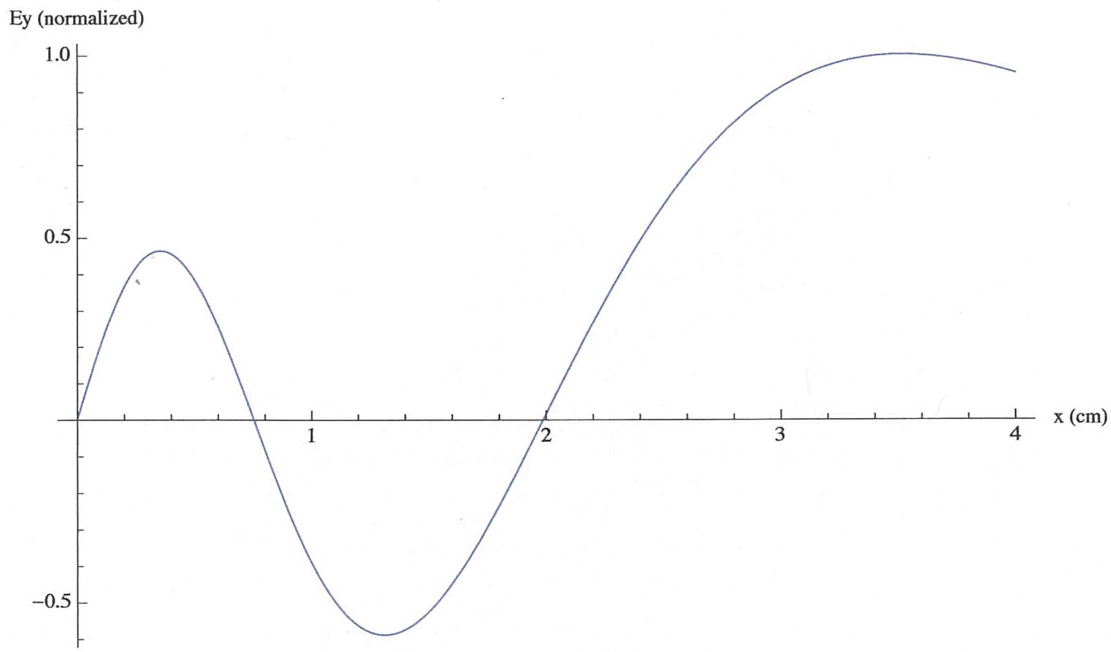


(a) Lowest-order mode.

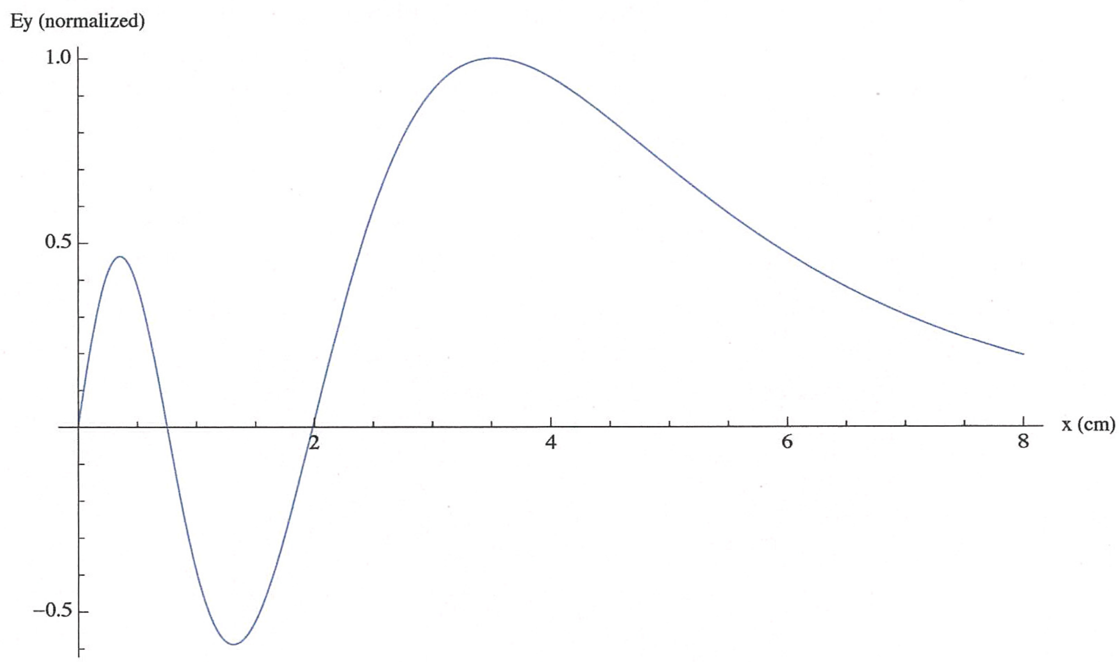


(b) Second lowest-order mode.

FIGURE 3. Normalized Electric Field, E_y, Versus x
for the TE Case $\left(\mu = b = \delta = 1,\ f_0 = 24 \text{ GHz}\right)$.



(c) Third lowest-order mode.



(d) Third lowest-order mode on larger scale.

FIGURE 3. (Contd.)

E_y is not exactly zero unless $x = \infty$, but we do not need to require this. We see that under the proper conditions, there is actually no need for the second conducting plate at all. The second boundary of the inhomogeneous region can be set at a value of x such that the magnitude of the field E_y is less than some prescribed amount, say $E_y \le 10^{-6}$ at $x = a'$.

Thus using Equations 10 and 18, after δ, b, μ and f_0 are assumed, one substitutes into Equation 10 (with the appropriate β from BC_1) giving

$$E_y\left(x = a'\right) = J_{\frac{2k_o\beta}{\delta}}\left(\frac{2k_0}{\delta}\sqrt{\mu b}\; e^{-\delta a'}\right) \tag{29}$$

By enforcing a prescribed value such as $E_y \le 10^{-6}$, we can calculate a value of a' (this is not the position of the second conducting plate). The second plate has been removed, and a' is the distance at which the field is appropriately small.

Figure 4 shows a plot of the normalized electric field (lowest-order mode) versus x for a higher frequency, $f_0 = 35$ GHz and $\mu = \delta = b = 1$. As frequency increases, the form of E_y becomes narrower and more closely confined to the plate at $x = 0$. As frequency increases, $E_y \le 10^{-6}$ will be enforced by using smaller and smaller values of a'. However, as the higher-order modes are considered (as in Figure 3), the higher the mode at a given frequency, the less tightly it is confined to $x = 0$ (see Figures 3(c) and 3(d)).

Figure 5 indicates what happens when $\mu = 1 = \delta$ but b is increased to 4 at $f_0 = 35$ GHz. By increasing either μ or b, one can confine the field even more tightly to $x = 0$ (note that Figure 5 only runs to $x = 2.0$, whereas Figures 3 and 4 run to $x = 4.0$).

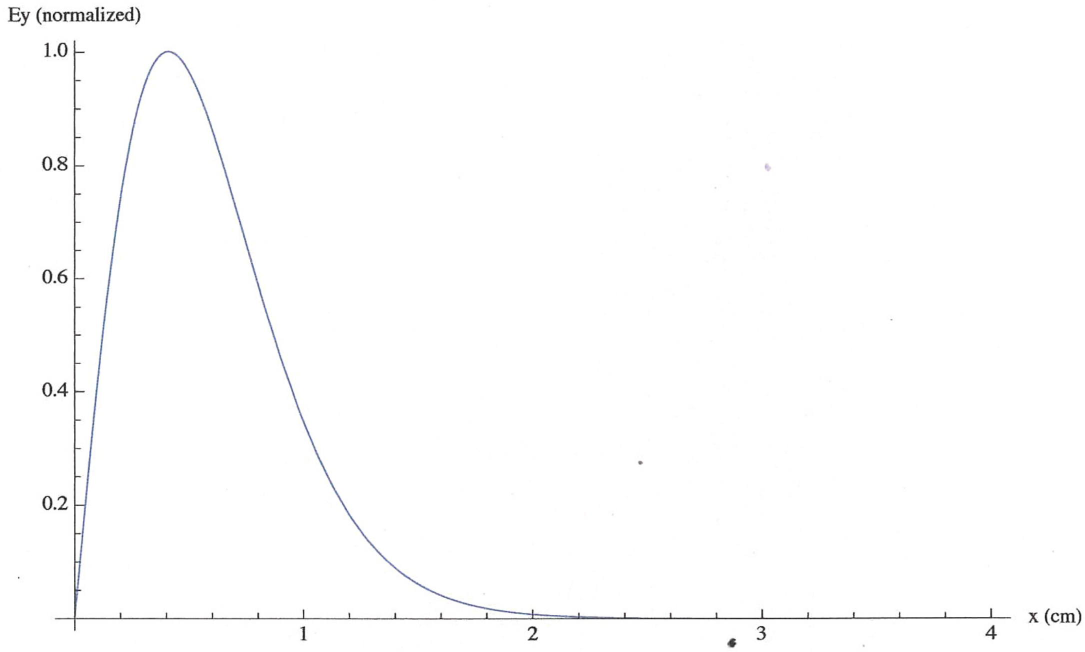


FIGURE 4. Normalized Electric Field, E_y, Versus x for the TE Case $\left(\mu = b = \delta = 1,\ f_0 = 35\text{ GHz}\right)$ (Lowest-Order Mode).

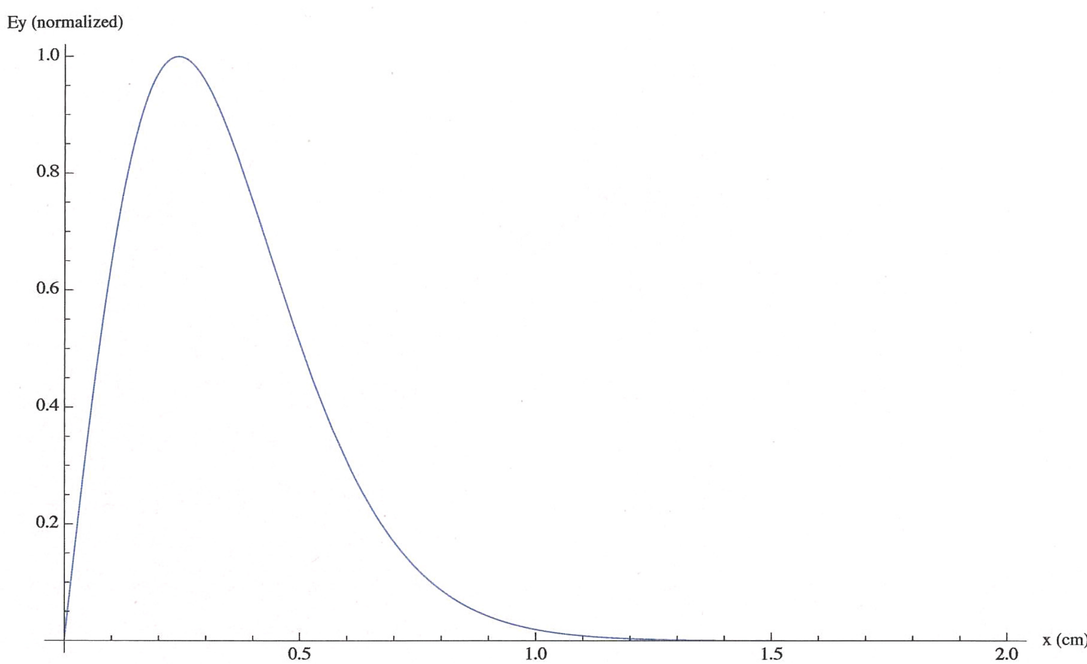


FIGURE 5. Normalized Electric Field, E_y, Versus x for the TE Case $\left(\mu = \delta = 1, b = 4,\ f_0 = 35\text{ GHz}\right)$ (Lowest-Order Mode).

The cutoff frequency for each mode occurs at $\beta = 0$ or from Equation 19,

$$f_{co}^{(\ell)} = \frac{j_{0,\ell} c\delta}{4\pi\sqrt{\mu b}}; \quad \ell = 1, 2, 3\ldots \tag{30}$$

where $j_{0,\ell}$ are the zeros of the zeroth order Bessel function. Thus (see Figure 2), the cutoff frequency of the lowest-order mode is given by the lowest zero of $J_0\left(j_{0,1}\right)$ (and with $\delta = 1 = \mu = b$)

$$f_{co}^{(1)} = 5.74 \text{ GHz}$$
$$f_{co}^{(2)} = 13.18 \text{ GHz}$$

and so on for the higher-order modes.

4.2 TM CASE

For the TM case, H_y is the only nonzero component of the magnetic field, and E_x and E_z are the corresponding nonzero electric field components. However, in this case there is no boundary condition that H_y (a tangential component) must obey. Thus, we work with E_z since E_z must be zero on both parallel plates. (As discussed in Section 3.0, we use D_z). In this case, Equation 28 is solved for β, the normalized propagation constant for the TM modes. In Table 2, the ordered TM roots, β, of Equation 28 are given for $\delta = 1$, $\sqrt{\mu b} = 1$, as a function of frequency. As the frequency increases, more and more higher-order modes can propagate assuming a fixed δ and $\sqrt{\mu b}$. Figure 6 is a plot of frequency versus β for the TM case. In general, the β's for the TM case are higher than those for the TE case at the same frequency. Figures 7(a), (b), and (c) show a plot of D_z (normalized) versus *x* in cm for the first three TM modes with $\delta = \mu = b = 1$ and $f_0 = 24$ GHz. The normalized D_z component is also tightly confined to the conducting plate at *x* = 0 and goes to 0 as *x* increases, again holding the TM fields within the inhomogeneous region.

TABLE 2. Ordered TM Roots, β, of Equation 28 With $\delta = 1$, $\sqrt{\mu b} = 1$.

f_0 (GHz)	β_1	β_2	β_3	β_4	β_5	β_6	β_7
6	0.1788						
7	0.3678						
8	0.4639						
9	0.5283						
10	0.5758						
11	0.6128						
12	0.6426						
13	0.6674						
14	0.6882	0.1281					
15	0.7061	0.1907					
16	0.7217	0.2367					
17	0.7354	0.2743					
18	0.7475	0.3064					
19	0.7584	0.3345					
20	0.7682	0.3595					
24	0.7993	0.4384	0.1706				
36	0.8526	0.5756	0.3782	0.2089	0.0302		
94	0.9263	0.7773	0.6719	0.5823	0.5022	0.4286	0.3599

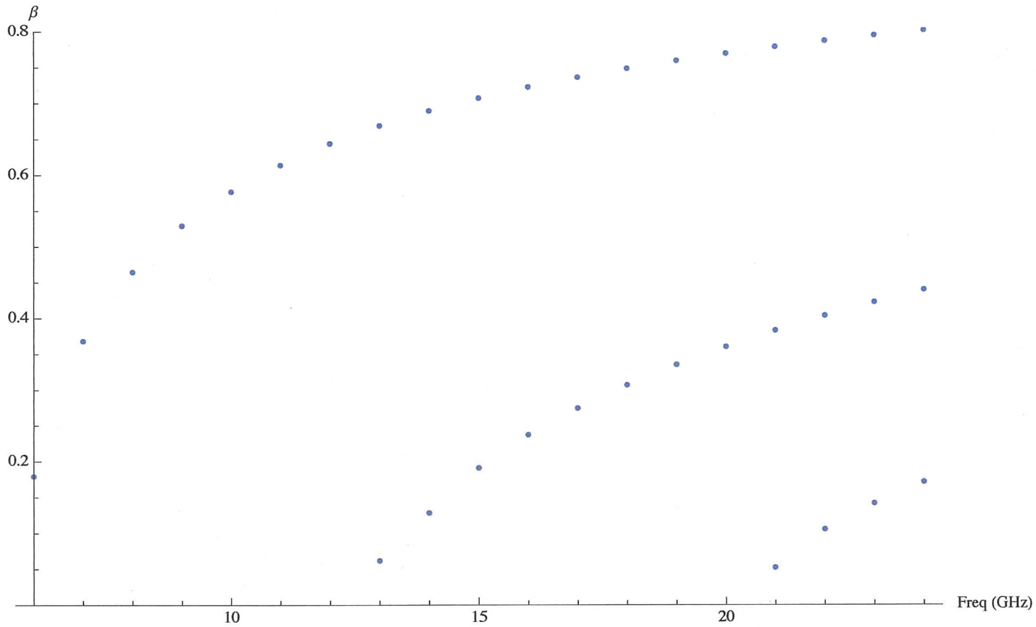
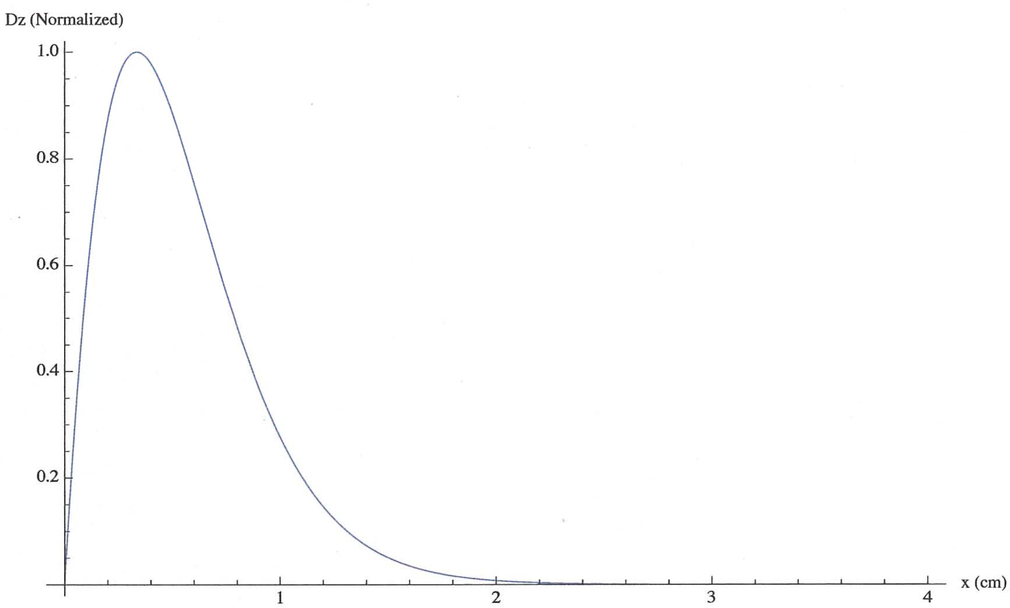
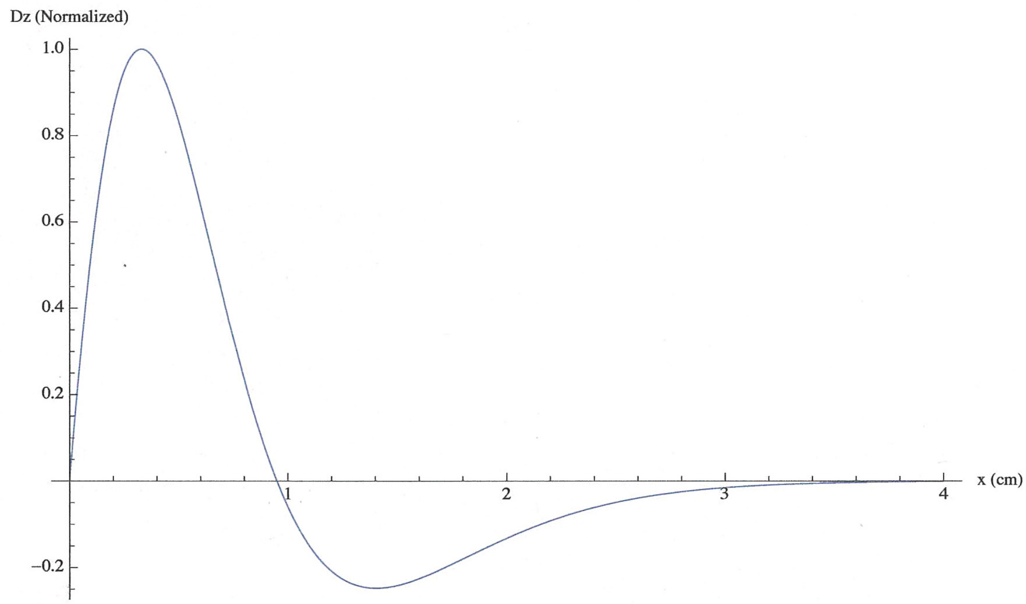


FIGURE 6. TM Longitudinal Propagation Constant, β,
Versus Frequency $(\mu = b = \delta = 1)$.

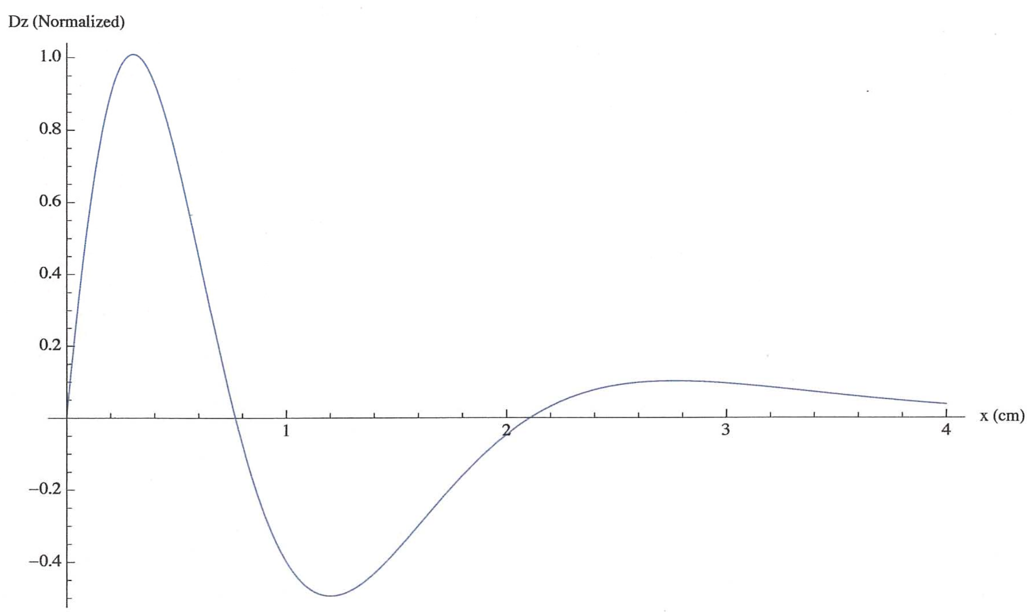


(a)

FIGURE 7. Normalized Electric Displacement, D_z Versus *x*.
Figures 7(a), (b), and (c) are the three lowest-order TM modes
$(\mu = b = \delta = 1,\ f_0 = 24\ \text{GHz})$.



(b)



(c)

FIGURE 7. (Contd.)

The TM mode cutoff frequencies are found when $\beta = 0$ or $\sigma = 1$. Using $\sigma = 1$ in Equation 28, Equation 28 becomes

$$\eta\left[J_0(\eta) - J_2(\eta)\right] + 2J_1(\eta) = 0 \tag{31}$$

Writing $J_2(\eta) = \dfrac{2J_1(\eta)}{\eta} - J_0(\eta)$, Equation 31 gives

$$J_0(\eta) = 0 \tag{32}$$

as the cutoff condition. Thus the TM mode cutoff condition is the same as that for the TE modes given by Equation 30. (This can also be seen numerically from Figures 2 and 6.)

5.0 COMPARISON OF INHOMOGENEOUS AND HOMOGENEOUS PARALLEL PLATE WAVEGUIDES

5.1 TE CASE

It is very interesting to compare the simple parallel plate waveguide containing a homogeneous medium and the parallel plate waveguide containing a medium with inhomogeneous permittivity in exponential form. The TE electric field in the homogeneous case is given by Reference 1

$$E_y = E_0 \sin\frac{m\pi x}{a} e^{ik_z z} e^{-i\omega t}; \quad m = 1, 2, 3\ldots \tag{33}$$

for the TE modes, while in the inhomogeneous case, it is given by Equation 10. E_y given by Equation 33 is a sinusoid that is periodic. E_y is zero at x = 0, but there must be a second conductor at x = a in order to have the field go to zero there. The higher order modes split the interval from x = 0 to a into equal subintervals. An infinite series of these guides could be placed along x = 0, a; x = a to 2a; x = 2a to 3a, and so on. The dispersion relation for the homogeneous parallel plate waveguide is given by

$$k_z^2 = k_0^2 \varepsilon\mu - \left(\frac{m\pi}{a}\right)^2 \tag{34}$$

A plot of k_z versus k_0 is shown in Reference 1 on page 135. It is pointed out that for the mth mode at some point, k_z will become imaginary if $k_0 < \dfrac{m\pi}{a\sqrt{\varepsilon\mu}}$. The wave then becomes evanescent and attenuates exponentially in the longitudinal direction.

Compare this with the dispersion relation for the inhomogeneous case

$$J_{\frac{2k_0\beta}{\delta}}\left(\frac{2k_0}{\delta}\sqrt{\mu b}\right) = 0 \tag{35}$$

$$\left(k_z = k_0\beta\right)$$

The homogeneous dispersion relation is explicitly known while the one above (Equation 35) is implicit. But while k_z in Equation 34 can become purely imaginary for $k_0 < \dfrac{m\pi}{a\sqrt{\varepsilon\mu}}$, if β is assumed to be imaginary in Equation 35, consider what happens.

Let $\beta = i\chi$, $\chi \in \mathbb{R}$ and then the dispersion relation becomes

$$J_{\frac{2k_0 i\chi}{\delta}}\left(\frac{2k_0}{\delta}\sqrt{\mu b}\right) = 0 \tag{36}$$

But this equation is complex, and while its real and imaginary parts do have zeros, they are not the same zeros. Thus there are no purely imaginary solutions $i\chi$ to the above (for μ, δ, b positive and real). For the inhomogeneous parallel plate waveguide, there are no evanescent waves that purely attenuate along $\hat{z}$ when μ, δ, and b are assumed to be positive and real. When both the order and the argument of Equation 35 are assumed to be complex, complex roots will occur (References 8 and 9). This is interesting in that for the TE case, the homogeneous guide containing a lossless isotropic medium has a denumerably infinite set of modes at any given frequency, most of which are evanescent and thus cut off with a few (or perhaps only one) propagating modes. The inhomogeneous waveguide containing a lossless isotropic inhomogeneous medium has a spectrum at any given frequency that is completely real and finite. At a given frequency (above the first cutoff) one has a finite number of propagating modes. Purely evanescent modes do not exist.

For the homogeneous parallel plate waveguide, the cutoff frequency is (References 1 and 10)

$$f_{co}^{(m)} = \frac{mc}{2a\sqrt{\varepsilon\mu}}; \quad m = 1, 2, 3, \ldots \tag{37}$$

where c is the speed of light. There are two ways to lower this cutoff frequency. One is by increasing ε and μ, and the other is by making a, the plate separation, larger. Similarly, when considering Equation 30, which gives the cutoff frequencies of the inhomogeneous parallel plate waveguide, the only way to lower the inhomogeneous cutoff frequencies is to either increase μ and b (analogous to increasing μ and ε in Equation 37) or to decrease the value of δ which governs the range of the permittivity. From Equation 29, decreasing δ implies that a' must be increased to give the prescribed value of $E_y \le 10^{-6}$ at $x = a'$.

5.2 TM CASE

In the TM case, the homogeneous parallel plate waveguide has a D_z component given by Reference 1

$$D_z = \frac{-H_0 i k_x}{\omega} \sin k_x x e^{i(k_z z - \omega t)} \tag{38}$$

where $k_x = \frac{m\pi}{a}$ (just as in the TE case except that m can be zero in the TM case). The D_z component in the inhomogeneous case is given by Equation 27. The TM dispersion relation for the homogeneous parallel plate waveguide is the same as for the TE case. The TM dispersion relation for the inhomogeneous guide is given by Equation 28.

Again, as with the TE case, k_z in Equation 34 can become purely imaginary for $k_0 < \frac{m\pi}{a\sqrt{\varepsilon\mu}}$. We look at Equation 28 to see what would happen if β were assumed to be purely imaginary. For $\beta = i\chi$, $\chi \in \mathbb{R}$,

$$\sigma = \sqrt{\left(\frac{2k_0 i\chi}{\delta}\right)^2 + 1} \tag{39}$$

or

$$\sigma = \sqrt{1 - \left(\frac{2k_0 \chi}{\delta} \right)^2} \tag{40}$$

Thus, while σ will still be real if $\frac{2k_0\chi}{\delta} < 1$, substitution of $\beta = i\chi$ into Equation 28 does not result in real solutions for χ. Although we have only considered a limited number of cases, no roots, χ, have been found that satisfy Equation 28. For $\frac{2k_0\chi}{\delta} > 1$, σ becomes imaginary and again no roots, χ, have been found that satisfy Equation 28 for μ, b, δ real and positive. (Thus, the inhomogeneous parallel plate waveguide spectrum again is completely real and finite, and there are no purely evanescent TM modes.) If any of μ, b, or δ is allowed to be complex, again complex roots can occur.

6.0 CONCLUSION

A parallel plate waveguide containing a region with a transverse exponential permittivity function of position has been analyzed. It has been shown that it is possible to eliminate one of the waveguide conducting plates and still keep the electromagnetic field closely confined to the remaining plate and held within the inhomogeneous region. A comparison between the inhomogeneous parallel plate geometry and the well-known homogeneous parallel plate waveguide shows a number of interesting differences.

First, for lossless media (i.e., μ, δ, b positive and real), the eigenvalue spectrum of the inhomogeneous parallel plate waveguide is real and finite at a given frequency as opposed to the spectrum of the homogeneous waveguide, which is denumerably infinite at a given frequency and composed of eigenvalues that are either purely real (propagating modes) or purely imaginary (attenuated modes). Thus, the inhomogeneous case has no purely attenuated modes.

Second, the eigenfunctions of the inhomogeneous case are not periodic sinusoids but are Bessel functions containing an exponential argument which decrease in amplitude as the transverse distance increases. While the eigenfunctions of the homogeneous parallel plate waveguide divide the interval between the plates into equidistant sections, the eigenfunctions of the inhomogeneous case can have nodes at any point along the interval from $x = 0$ to $x = a'$, and the number of nodes is small. All fields in the transverse direction attenuate exponentially. This sharp amplitude decrease allows the removal of the second conducting plate and causes the field components to be closely confined to the remaining plate.

Third, the cutoff frequencies for the inhomogeneous case are dependent on the zeros of the zeroth order Bessel function, while the cutoff frequencies for the homogeneous case are dependent on the set of positive integers.

All numerical results have been based on the assumption of lossless media. In future work, lossy media (μ and b complex) will be considered. Also, δ complex will be assumed, which changes the purely exponentially attenuating permittivity function into one that is sinusoidally oscillating and damped.

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Appendix A

ASYMPTOTIC FORM OF E_y AND H_y

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For the TE case, E_y is given by Equations 10 and 11. For large *x*, the argument in Equation 10 becomes very small, and $J_\nu\left(\eta e^{-\delta x/2}\right)$ can be approximated by the first term of its series expansion. Thus for large *x*,

$$E_y(x) \approx c_1 \left(\frac{\eta e^{-\delta x/2}}{2} \right)^\nu \tag{A-1}$$

or

$$E_y(x) \approx c_1 \left(\frac{\eta}{2} \right)^\nu e^{-\delta \nu x/2} \tag{A-2}$$

The total field becomes

$$E_y^{total} \approx c_1 \left(\frac{k_0 \beta}{\delta} \right)^\nu e^{(-k_0 \beta x + i k_0 \beta z - i\omega t)} \tag{A-3}$$

where β, η, ν are assumed to be both positive and real.

For the TM case, for large *x*,

$$H_y(x) \approx d_1 \left(\frac{k_0 \beta}{\delta} \right)^{1+\sigma} e^{-\delta(1+\sigma)x/2} \tag{A-4}$$

where $\sigma = \sqrt{\nu^2 + 1}$ and η, and σ are given by Equation 22 and previously. The total asymptotic field

$$H_y^{total} \approx d_1 \left(\frac{k_0 \beta}{\delta} \right)^{1+\sigma} e^{-\left[\delta(1+\sigma)\frac{x}{2} - i k_0 \beta z + i\omega t\right]} \tag{A-5}$$

For large *x*, all field components fall off as exponentials.

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Appendix B

ORTHOGONALITY OF FIELD COMPONENTS FOR THE INHOMOGENEOUS PERMITTIVITY $\varepsilon(x) = be^{-\delta x}$

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In the TE case, the electric field has only a *y* component,

$$E_y = c_1' J_\nu \left(\eta e^{-\delta x/2}\right) \tag{B-1}$$

with ν and η given in Section 3.0.

For large enough frequencies, it is possible to have more than one β value satisfying the boundary condition at $x = 0$ given by Equation 18. When k_0, b, μ, δ are real, positive, and have fixed values, if there are two solutions to Equation 18 (i.e., β_1 and β_2), then the corresponding eigenfunctions are

$$E_y^{(1)} = c_1'^{(1)} J_{\nu_1} \left(\eta e^{-\delta x/2}\right) \tag{B-2a}$$

with $\nu_1 = \dfrac{2k_0\beta_1}{\delta}$, $\eta = \dfrac{2k_0\sqrt{b\mu}}{\delta}$

and

$$E_y^{(2)} = c_1'^{(2)} J_{\nu_2} \left(\eta e^{-\delta x/2}\right) \tag{B-2b}$$

with $\nu_2 = \dfrac{2k_0\beta_2}{\delta}$, η as above.

$E_y^{(1)}$ and $E_y^{(2)}$ are orthogonal if

$$\int_0^\infty E_y^{(1)}(x) E_y^{(2)}(x) w(x)\, dx = 0 \tag{B-3}$$

where $w(x)$ is some weighting function.

For now we set $w(x) = 1$.

Let

$$I = \int_0^\infty E_y^{(1)}(x) E_y^{(2)}(x)\, dx = c_1'^{(1)} c_1'^{(2)} \int_0^\infty J_{\nu_1}\left(\eta e^{-\delta x/2}\right) J_{\nu_2}\left(\eta e^{-\delta x/2}\right) dx \tag{B-4}$$

Making the variable change,

$$u = e^{-\delta x/2} \tag{B-5}$$

we obtain

$$dx = -\frac{2}{\delta}\frac{du}{u} \tag{B-6}$$

Substitution of Equations B-5 and B-6 into Equation B-4 gives

$$I = \left(\frac{2}{\delta}\right) c_1'^{(1)} c_1'^{(2)} \int_0^1 \frac{J_{\nu_1}(\eta u) J_{\nu_2}(\eta u)\, du}{u} \tag{B-7}$$

Upon integration, *I* becomes

$$I = \frac{\frac{2}{\delta} c_1'^{(1)} c_1'^{(2)}}{\left(\nu_1^2 - \nu_2^2\right)} \left\{ u\eta J_{\nu_1 - 1}(u\eta) J_{\nu_2}(u\eta) - J_{\nu_1}(u\eta)\left[u\eta J_{\nu_2 - 1}(u\eta) + (\nu_1 - \nu_2) J_{\nu_2}(u\eta)\right]\right\}\Bigg|_0^1 \tag{B-8}$$

Since $J_{\nu_1}(\eta) = 0$, and $J_{\nu_2}(\eta) = 0$ from the boundary condition, upon evaluation at the limits, *I* reduces to zero. Thus, two eigenfunctions with different β's at the same frequency are orthogonal. The TM case is more complicated but done similarly.

If the orders of the eigenfunctions are different and the arguments are different also, orthogonality does not hold.

Appendix C

THE POYNTING VECTOR AND POWER FLOW

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TE CASE

The complex Poynting vector is

$$\vec{S} = \vec{E}\times\vec{H}^* \tag{C-1}$$

$$= E_y\hat{y}\times\left[H_x^*\hat{x}+H_z^*\hat{z}\right] \tag{C-2}$$

$$= -E_yH_x^*\hat{z}+E_yH_z^*\hat{x} \tag{C-3}$$

where * implies complex conjugate. E_y is given by Equation 12, H_x and H_z by Equation 14. With μ, *b*, δ, k_0, and β all real and positive

$$E_y = c_1' J_\nu\left(\eta e^{-\delta x/2}\right), \tag{C-4}$$

$$H_x^* = \frac{-k_0\beta}{\omega\mu_0\mu}c_1' J_\nu\left(\eta e^{-\delta x/2}\right) \tag{C-5}$$

and H_z^* is given by the complex conjugate of Equation 15. (The *z* and time dependencies have cancelled.)

From Equations C-3, C-4, and C-5, we see that the *z* component of the Poynting vector is real, i.e.,

$$S_z = \frac{c_1'^2 k_0\beta}{\omega\mu_0\mu}J_\nu{}^2\left(\eta e^{-\delta x/2}\right) \tag{C-6}$$

while the *x* component of the Poynting vector is imaginary.

Because the power is defined as

$$P = Re\iint_{cross\ section}\left(\vec{E}\times\vec{H}^*\right)\cdot d\bar{z} \tag{C-7}$$

the power flowing in the $\hat{z}$-direction is

$$P = \frac{c_1'^2 k_0\beta}{\omega\mu_0\mu}\int_0^\infty J_\nu^2\left(\eta e^{-\delta x/2}\right)dx \tag{C-8}$$

or using the same variable substitution as in Appendix B,

$$P = \frac{2c_1'^2 k_0 \beta}{\delta \omega \mu_0 \mu} \int_0^1 \frac{J_\nu^2(\eta u)}{u}\, du \tag{C-9}$$

Upon integration the power becomes

$$P = \frac{c_1'^2 k_0 \beta}{\delta \omega \mu_0 \mu} \frac{1}{\nu \left[\Gamma(1+\nu)\right]^2} \left\{ \left(\frac{\eta}{2}\right)^{2\nu} {}_2F_3\left[\left\{\nu, \frac{1}{2}+\nu\right\}, \left\{1+\nu, 1+\nu, 1+2\nu\right\}, -\eta^2\right] \right\} \tag{C-10}$$

where ${}_2F_3$ is a generalized hypergeometric function with two numeratorial parameters and three denominatorial ones as defined in Mathematica.

TM CASE

In this case, the complex Poynting vector is

$$\vec{S} = E_x H_y^* \hat{z} - E_z H_y^* \hat{x} \tag{C-11}$$

Assuming all variables real and positive, H_y^* is given by Equation 24 and E_x and E_z by Equation 25. The *z* component of the Poynting vector is real and given by

$$S_z = \frac{k_0 \beta}{\omega \varepsilon_0 \varepsilon(x)} \left(\frac{\eta}{2}\right)^2 d_1'^2 e^{-\delta x} J_\sigma^2\left(\eta e^{-\delta x/2}\right) \tag{C-12}$$

or

$$S_z = \frac{k_0 \beta}{\omega \varepsilon_0 b} \left(\frac{\eta}{2}\right)^2 d_1'^2 J_\sigma^2\left(\eta e^{-\delta x/2}\right) \tag{C-13}$$

Using Equation C-7 and the same procedure as in the TE case, the power for a TM mode is

$$P = \frac{k_0 \beta d_1'^2}{\omega \varepsilon_0 b} \left(\frac{\eta}{2}\right)^2 \frac{1}{\sigma\left[\Gamma(1+\sigma)\right]^2} \left\{ \left(\frac{\eta}{2}\right)^{2\sigma} {}_2F_3\left[\left\{\sigma, \frac{1}{2}+\sigma\right\}, \left\{1+\sigma, 1+\sigma, 1+2\sigma\right\}, -\eta^2\right] \right\} \tag{C-14}$$

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